

# Strict Deformation Quantization for Actions of a Class of Symplectic Lie Groups

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## Abstract

We present explicit universal strict deformation quantization formulae for actions of Iwasawa subgroups  $AN$  of  $SU(1, n)$ . This answers a question raised by Rieffel in [13].

## Introduction

In [12], Rieffel describes explicitly strict quantization for actions of  $\mathbb{R}^d$ . Roughly, the idea is as follows. Let  $A$  be an associative (topological) algebra and let  $\alpha : \mathbb{R}^d \times A \rightarrow A$  be an action of  $\mathbb{R}^d$  on  $A$  by automorphisms. Then, if the situation is regular enough, one can give a sense to the following product :

$$a \star_J b \stackrel{\text{def.}}{=} \int \alpha_x(a) \alpha_{Jy}(b) e^{i\langle x, y \rangle} dx dy \quad (1)$$

as an oscillatory integral. In the previous formula,  $a$  and  $b$  are elements of  $A$ ,  $\langle \cdot, \cdot \rangle$  is a Euclidean scalar product on  $\mathbb{R}^d$  and  $J$  is a skewsymmetric matrix  $J \in so(\mathbb{R}^d, \langle \cdot, \cdot \rangle)$ . Also,  $dx dy$  denotes a Haar measure on  $\mathbb{R}^d \times \mathbb{R}^d$ . The product  $\star_J$  is a deformation of the product on  $A$ . Indeed, when  $J$  tends to 0 in  $so(\mathbb{R}^d, \langle \cdot, \cdot \rangle)$ ,  $\star_J$  tends to the product on  $A$ . Moreover, the product  $\star_J$  turns out to be associative. In the case  $M$  is some manifold  $\mathbb{R}^d$  acts on by diffeomorphisms, one gets an action on  $A = C_\infty(M)$ . Therefore, introducing a deformation parameter  $\hbar$  multiplying  $J$ , the above construction yields a deformation quantization of the Poisson bracket on  $M$  induced by the action and the data of  $J$ . Formula (1) is universal in the sense that it is valid for any action of  $\mathbb{R}^d$ .

An approach to universal deformation formulae (UDF) has been proposed by Giaquinto and Zhang [6] within the formal framework i.e. the resulting deformation is a formal power series, such as a star product for instance. Among other things, Giaquinto and Zhang present there a beautiful explicit formal UDF for actions of the (non-Abelian) Lie group “ $ax + b$ ” of affine transformations of the real line. As observed by Rieffel in [13], this suggests that, at least for the group “ $ax + b$ ”, there should exist UDF’s at the analytical (strict) level.

It has been observed in [4] that a possible approach to “universal” deformations for (non Abelian) group actions is to study a particular class of three-point kernels on group manifolds. Roughly speaking, this means the following. Let  $G$  be a Lie group endowed with a left-invariant Haar measure  $\mu$ . Assume the existence of a (non trivial) function space  $\mathcal{E} \subset C(G)$  and a three-point kernel  $K \in C(G \times G \times G)$  such that

- (i)  $K$  is invariant under the diagonal left action of  $G$  on  $G \times G \times G$ ;

(ii) for all  $u, v \in \mathcal{E}$ , the formula

$$u \star^G v(x) = \int_{G \times G} u(g) v(h) K(x, g, h) \mu_g \mu_h$$

defines an element  $u \star^G v$  of  $\mathcal{E}$ <sup>1</sup>;

(iii)  $(\mathcal{E}, \star^G)$  is an associative algebra.

Then, if an action  $\alpha : G \times A \rightarrow A$  of  $G$  on some (topological) algebra  $A$  is given, one checks that formally the formula

$$a \star^A b \stackrel{\text{def.}}{=} \int_{G \times G} K(e, g, h) \alpha_g(a) \alpha_h(b) \mu_g \mu_h, \quad (2)$$

where  $e$  is the unit element of  $G$ , defines an associative product on  $A$ , provided some regularity condition is fulfilled (e.g. the function  $G \rightarrow A : g \rightarrow \alpha_g(a)$  has the same type of regularity as the elements of  $\mathcal{E}$ ).

For example, in the case of the Abelian group  $G = \mathbb{R}^d$ , Rieffel's product (1) coincides with (2) when

$$K(x, y, z) = \exp \left( \frac{2i}{\hbar} \{ \langle x, Jy \rangle + \langle y, Jz \rangle + \langle z, Jx \rangle \} \right)$$

and when  $\mathcal{E} = C_\infty(\mathbb{R}^d)$ .

In this paper, we explicitly describe universal deformation three-point kernels  $K$  as above for a class of solvable Lie groups (see Theorem 3.1). Those are Iwasawa  $AN$  subgroups of  $SU(1, n)$ . In particular, any action of  $SU(1, n)$  yields an action of  $AN$ . This should provide an interesting class of strict deformation quantizations of (non regular) Poisson manifolds such as quantum flag manifolds. However, this last point will not be investigated in the present article (see nevertheless Section 5).

This note is organized as follows.

## 1 Rank one non-compact Hermitian symmetric spaces

Let  $G = SU(1, n)$  and  $R = AN$  be an Iwasawa subgroup, i.e.  $R$  is the  $AN$  factor in an Iwasawa decomposition  $G = ANK$  of  $G$ . Then the group manifold  $R$  is equivariantly diffeomorphic to the Hermitian symmetric space  $M = G/K$ . In particular, the Lie group  $R$  carries an  $R$ -left-invariant symplectic structure  $\omega$ . In this section, we describe a global Darboux chart on  $(R, \omega)$  which will be important later on.

## 2 Guessing the deformed product formula

The proof of our main result (Theorem 3.1) does not depend on the present section. Nevertheless, it explains how Formula (14) has been found.

### 2.1 $AN$ -covariant Moyal star products

The main property of our Darboux chart on  $(R, \omega)$  (cf. Section 1) is that, with respect to this chart, the formal Moyal star product is covariant in Arnal's sense (see Definition 2.2). One can therefore apply techniques of star-representation theory [2, 1, 5] to analyze the star-representation of  $R$  arising from the covariance property.

### 2.2 The $\mathcal{Z}_\hbar$ transform

Using the cocycle defining the star representation, we introduce some kind of Fourier integral operator (the  $\mathcal{Z}_\hbar$  transform) which intertwines the pointwise commutative product on  $C^\infty(R)[[\hbar]]$  with an  $\hbar$ -dependent (commutative) product with respect to which the Lie algebra of  $R$  acts by derivations. The “commutative manifold” underlying the latter product therefore carries an  $R$ -invariant deformation quantization. This  $R$ -space turns out to be formally  $R$ -equivariantly isomorphic to the group  $R$ .

## 3 Strict Quantization

We define a one parameter family of (Fréchet) function spaces  $\{\mathcal{E}_\hbar\}_{\hbar \geq 0}$ , each of them endowed with an associative product  $\star_\hbar$ , such that

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<sup>1</sup>This condition is too strong in general. The product formula should hold only on a dense subset of  $\mathcal{E}$ , provided a topological framework is defined (see e.g. [12] or Theorem 3.1).

(i) for all  $\hbar \geq 0$ , one has a dense inclusion

$$C_c^\infty(R) \subset \mathcal{E}_\hbar;$$

(ii) for all  $u, v \in C_c^\infty(R)$ , the product reads

$$u \star_\hbar v(x) = \hbar^{-2 \dim R} \int_{R \times R} e^{\frac{i}{\hbar} S(x, y, z)} a(x, y, z) u(y) v(z) dy dz$$

where  $K = a e^{\frac{i}{\hbar} S}$  is an  $R$ -left-invariant three-point kernel on  $R \times R \times R$ , and where  $dy dz$  is a left invariant Haar measure on  $R \times R$  (see Theorem 3.1).

#### 4 The case $n = 1$

We relate the present construction in the case  $n = 1$  with a strict quantization of the symplectic symmetric space  $(SO(1, 1) \times \mathbb{R}^2)/\mathbb{R}$  obtained in [4]. This leads to a relation between our work and Unterberger's composition formula for the one-dimensional Klein-Gordon calculus [14, 15].

#### 5 Remark for further developments

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## 1 Rank one non-compact Hermitian symmetric spaces

### 1.1 Notations and preliminaries

We refer to [8] for the general theory of (Riemannian) symmetric spaces.

Let  $G$  be a connected real **simple Lie group** with Lie algebra  $\mathcal{G}$ . We denote by  $B$  the **Killing form** on  $\mathcal{G}$ . Let  $\sigma$  be a Cartan involution of  $\mathcal{G}$  with Cartan decomposition

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{P} \quad (\sigma = id_{\mathcal{K}} \oplus (-id_{\mathcal{P}})).$$

Fix a maximal Abelian subalgebra  $\mathcal{A}$  contained in  $\mathcal{P}$ . The dimension of  $\mathcal{A}$  is called the (real) **rank** of  $\mathcal{G}$ . Let  $\Phi$  be the **root system** of  $\mathcal{G}$  with respect to  $\mathcal{A}$  and fix a positive root system  $\Phi^+$  ( $\Phi = \Phi^+ \cup (-\Phi^+)$ ). Denote by  $\mathcal{G}_\alpha$  the **weight space** corresponding to  $\alpha \in \Phi \cup \{0\}$ . One then has

$$\mathcal{G} = \sigma(\mathcal{N}) \oplus \mathcal{G}_0 \oplus \mathcal{N} \tag{3}$$

with

$$\mathcal{N} = \sum_{\alpha \in \Phi^+} \mathcal{G}_\alpha \text{ and } \mathcal{G}_0 = \mathcal{Z}_{\mathcal{K}}(\mathcal{A}) \oplus \mathcal{A}$$

where  $\mathcal{Z}_{\mathcal{K}}(\mathcal{A})$  denotes the centralizer of  $\mathcal{A}$  in  $\mathcal{K}$ . One also has the **Iwasawa decomposition**

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{A} \oplus \mathcal{N}$$

which induces a global analytic diffeomorphism :

$$A \times N \times K \rightarrow G : (a, n, k) \rightarrow ank$$

where  $K$  (resp.  $A$ , resp.  $N$ ) denotes the (connected) analytic subgroup of  $G$  with algebra  $\mathcal{K}$  (resp.  $\mathcal{A}$ , resp.  $\mathcal{N}$ ). The Iwasawa group decomposition therefore induces a global diffeomorphism between the group manifold  $R = AN$  and the Riemannian symmetric space  $M = G/K$  :

$$R \xrightarrow{\sim} G/K : (a, n) \rightarrow anK. \tag{4}$$

Observe that the vector space  $\mathcal{P} \subset \mathcal{G}$  is naturally identified with the tangent space  $T_K(M)$ .

**Definition 1.1** *The symmetric space  $M = G/K$  is **Hermitian** if there exists an endomorphism  $J \in \text{End}(\mathcal{P})$  of the vector space  $\mathcal{P}$  such that :*

- (i)  $J^2 = -\text{id}_{\mathcal{P}}$
- (ii)  $B(JX, JY) = B(X, Y) \quad \forall X, Y \in \mathcal{P}$
- (iii)  $\text{ad}(k) \circ J = J \circ \text{ad}(k) \quad \forall k \in \mathcal{K}$ .

In this case, the tensors  $J$  and  $B$  on  $\mathcal{P} = T_K(M)$  globalize to  $M$  respectively as a complex structure  $\mathcal{J}$  and a Riemannian metric  $\mathbf{g}$  on  $M$ . Moreover, the 2-form  $\omega$  on  $M$  defined by

$$\omega_x(X, Y) = \mathbf{g}_x(\mathcal{J}X, Y) \quad x \in M; X, Y \in T_x(M)$$

is a  $G$ -invariant symplectic structure on  $M$ .

## 1.2 (Co)Adjoint orbits

We now realize our Hermitian symmetric space  $M = G/K$  as a coadjoint orbit in  $\mathcal{G}^*$  or equivalently, using the Killing form  $B$ , as an adjoint orbit in  $\mathcal{G}$ .

Let us first denote by  $\Omega \in \bigwedge^2(\mathcal{G}^*)$  the skewsymmetric 2-form on  $\mathcal{G}$  defined by

$$\begin{aligned} \Omega(X, Y) &= B(JX, Y) \quad X, Y \in \mathcal{P}; \\ \Omega(\mathcal{K}, \mathcal{G}) &= 0. \end{aligned}$$

One observes, using the  $\mathcal{K}$ -invariance, that the 2-form  $\Omega$  is a Chevalley 2-cocycle for the trivial representation of  $\mathcal{G}$  on  $\mathbb{R}$  :

$$\Omega \in C_{\text{Chevalley}}^2(\mathcal{G}, \mathbb{R}); \quad \delta\Omega = 0.$$

Whitehead's lemmas then tell us that there exists an element  $\xi_0 \in \mathcal{G}^*$  such that

$$\delta\xi_0 = \Omega. \tag{5}$$

Equivalently, one gets  $Z_0 \in \mathcal{G}$  such that

$$B(Z_0, \cdot) = \xi_0.$$

Observe that the definition of  $\Omega$  implies

- (i)  $Z_0 \in \mathcal{Z}(\mathcal{K})$ ,
- (ii)  $J = \pm \text{ad}(Z_0)|_{\mathcal{P}}$

where  $\mathcal{Z}(\mathcal{K})$  denotes the center of  $\mathcal{K}$ . This, together with a little more work (see [3, 9]) provides the following ‘‘Hamiltonian’’ description of Hermitian symmetric spaces according to Kostant's classification of homogeneous Hamiltonian spaces.

**Proposition 1.1** *(i) A symmetric space  $G/K$  is Hermitian if and only if  $\mathcal{Z}(\mathcal{K}) \neq 0$ .*

*(ii) In this case,  $\dim(\mathcal{Z}(\mathcal{K})) = 1$  and the map*

$$\mathcal{Z}(\mathcal{K}) \setminus \{0\} \rightarrow \bigwedge^2(\mathcal{P}^*) : Z \rightarrow \delta B(Z, \cdot)|_{\mathcal{P} \times \mathcal{P}}$$

*induces a bijection onto the set of  $\mathcal{K}$ -invariant bilinear symplectic forms on  $\mathcal{P}$ .*

*(iii) The Hermitian symmetric space  $(M, \omega, \mathcal{J})$  is then realized as the adjoint orbit  $\mathcal{O} = \text{Ad}(G)Z \subset \mathcal{G}$  of  $Z$  via*

$$M = G/K \xrightarrow{\sim} \mathcal{O} : gK \rightarrow \text{Ad}(g)Z.$$

*Under this identification, the symplectic form  $\omega$  corresponds to the **Kostant symplectic form**  $\omega^{\mathcal{O}}$  on  $\mathcal{O}$  defined by*

$$\omega_x^{\mathcal{O}}(X^*, Y^*) = -B(x, [X, Y]) \tag{6}$$

*where  $x \in \mathcal{O} \subset \mathcal{G}$ ;  $X, Y \in \mathcal{G}$  and where  $X^*$  denotes the fundamental vector field associated to  $X \in \mathcal{G}$  on  $\mathcal{O}$  :*

$$X_x^* = \frac{d}{dt}\bigg|_0 \text{Ad}(\exp(-tX))x.$$

### 1.3 A class of Kählerian groups and their Iwasawa coordinates

When in a Hermitian situation, the diffeomorphism (4) endows the group manifold  $R = AN$  with the transported symplectic form  $\omega$  coming from  $M$ . The symplectic form  $\omega$  on  $R$  is then invariant under the left action

$$L : R \times R \rightarrow R : (x, y) \rightarrow xy = L_x y.$$

A Lie group with a left invariant symplectic structure is called a **symplectic group** [10].

**Proposition 1.2** *Let  $M = G/K$  be an irreducible Hermitian symmetric space of the non-compact type. Let  $(R = AN, \omega)$  be the associated symplectic group via the isomorphism (4). Denote by  $\mathcal{R} = \mathcal{A} \oplus \mathcal{N}$  its Lie algebra. Then, the map*

$$\mathcal{R} = \mathcal{A} \oplus \mathcal{N} \xrightarrow{\mathcal{I}} R : (a, n) \rightarrow \exp(a) \exp(n)$$

*is a global diffeomorphism called **Iwasawa coordinates**. Moreover, through the map  $\mathcal{I}$ , the symplectic form reads*

$$(\mathcal{I}^* \omega)_r(u, v) = B(Z, [Ad(\exp(-n))u_{\mathcal{A}} + F(ad(n))u_{\mathcal{N}}, (u \leftrightarrow v)])$$

*where  $r = (a, n) \in \mathcal{R} \subset \mathcal{G}$  and  $u, v \in T_r(\mathcal{R}) = \mathcal{R} \subset \mathcal{G}$ , where we write  $u = u_{\mathcal{A}} + u_{\mathcal{N}}$  according to the decomposition  $\mathcal{R} = \mathcal{A} \oplus \mathcal{N}$  and where  $F$  is the analytic function defined by*

$$F(z) = \frac{1 - e^{-z}}{z} \quad (z \in \mathbb{C}).$$

*Proof.* Let  $Z \in \mathcal{Z}(\mathcal{K})$  be as in Section 1.2 and denote by  $\varphi : \mathcal{R} \rightarrow \mathcal{O} = Ad(G)Z \subset \mathcal{G}$  the global diffeomorphism defined by

$$\varphi(r) = Ad(\mathcal{I}(r))Z.$$

Identifying  $T_r \mathcal{R}$  with  $\mathcal{R} \subset \mathcal{G}$ , one has for  $u \in T_r \mathcal{R}$  :

$$\begin{aligned} \varphi_{*r}(u) &= \frac{d}{dt} \Big|_0 Ad(\exp(a + tu_{\mathcal{A}}) \exp(n + tu_{\mathcal{N}}))Z = \\ &= -u_{\mathcal{A}}^*|_{\varphi(r)} + Ad(\exp a) \frac{d}{dt} \Big|_0 Ad(\exp n) Ad(\exp -n) Ad(\exp(n + tu_{\mathcal{N}}))Z = \\ &= -u_{\mathcal{A}}^*|_{\varphi(r)} + Ad(\mathcal{I}(r)) \frac{d}{dt} \Big|_0 Ad(CBH(-n, n + tu_{\mathcal{N}}))Z \end{aligned}$$

where  $CBH$  is the Campbell-Backer-Hausdorff function for the group  $N$  ( $\exp x \cdot \exp y = \exp CBH(x, y)$ ). Now, since

$$\frac{d}{dt} \Big|_0 CBH(-n, n + tu_{\mathcal{N}}) = F(ad(n))u_{\mathcal{N}} \quad (\text{see [8]}),$$

one gets

$$\varphi_{*r}(u) = -u_{\mathcal{A}}^*|_{\varphi(r)} - (Ad(\mathcal{I}(r))F(ad(n))u_{\mathcal{N}})^*_{\varphi(r)}. \quad (7)$$

Hence, using the  $Ad$ -invariance of the Killing form and the fact that  $Ad(\exp a)|_{\mathcal{A}} = id_{\mathcal{A}}$ , Formula (6) yields the result.  $\blacksquare$

On every Hermitian symmetric space of the non-compact type  $M$ , there actually exists a global **Darboux chart** i.e. a coordinate system where the symplectic structure reads constantly. Indeed, one can realize  $M$  as a coadjoint orbit of  $R$  : the orbit of the element  $\iota^* \xi_0$  where  $\mathcal{R} \xrightarrow{\iota} \mathcal{G}$  is the canonical injection. A result of Pedersen [11] then states that on the universal covering of every coadjoint orbit of a solvable Lie group there exists a global Darboux chart.

We will show that, at least in the rank one case, the Iwasawa coordinates explicitly yield such a global Darboux chart. Before this, we establish the following lemma which will be useful further on.

**Lemma 1.1** *Let  $\mathcal{G}$  be a simple Lie algebra with Iwasawa decomposition  $\mathcal{G} = \mathcal{K} \oplus \mathcal{A} \oplus \mathcal{N}$ . Assume  $\mathcal{Z}(\mathcal{K}) \neq 0$ . Then,*

$$\dim \mathcal{A} \geq \dim \mathcal{Z}(\mathcal{N})$$

*where  $\mathcal{Z}(\mathcal{N})$  denotes the center of  $\mathcal{N}$ .*

*Proof.* Let  $\mathcal{R} \xrightarrow{\iota} \mathcal{G}$  be the canonical inclusion and let  $\xi_0 \in \mathcal{G}^*$  be such that  $\delta\xi_0 = \Omega$  (cf. (5)). Then, since  $\mathcal{K} \cap \mathcal{R} = 0$ , the radical of  $\delta\iota^*\xi_0$  in  $\mathcal{R}$  is trivial. Moreover, if  $V$  denotes the radical of  $\Omega$  in  $\mathcal{N}$ , one has  $\mathcal{Z}(\mathcal{N}) \subset V$ . Indeed, if  $z \in \mathcal{Z}(\mathcal{N})$ , one has  $\Omega(\mathcal{N}, z) = \xi_0[z, \mathcal{N}] = 0$ . Observe now that the map

$$V \rightarrow \mathcal{A}^* : v \rightarrow \Omega(v, \cdot)|_{\mathcal{A}}$$

is injective. Indeed, let  $v \in V$  be such that  $\Omega(v, \mathcal{A}) = 0$ . Then,  $0 = \Omega(v, \mathcal{A} \oplus \mathcal{N}) = \delta\iota^*\xi_0(v, \mathcal{R})$  hence  $v = 0$ . Thus  $\dim \mathcal{A}^* = \dim \mathcal{A} \geq \dim V \geq \dim \mathcal{Z}(\mathcal{N})$ .  $\blacksquare$

## 1.4 Rank one

**Proposition 1.3** *Let  $M = G/K$  and  $(R = AN, \omega)$  be as in Proposition 1.2. Assume  $\dim \mathcal{A} = 1$ . Then, the Iwasawa coordinates  $\mathcal{R} \xrightarrow{\mathcal{I}} R$  define a global Darboux chart on  $(R = AN, \omega)$  i.e.  $\mathcal{I}^*(\omega)$  is a constant bilinear 2-form on the vector space  $\mathcal{R}$ .*

Before passing to the proof, we first recall the following classical result about the structure of  $\mathcal{R}$ .

**Lemma 1.2** *Assume  $\dim \mathcal{A} = 1$  and  $\dim \mathcal{G} > 3$ . Then*

- (i)  $\Phi = \{\pm\alpha, \pm 2\alpha\}$ ;
- (ii)  $\mathcal{N} = \mathcal{G}_\alpha \oplus \mathcal{G}_{2\alpha}$  and  $\mathcal{G}_{2\alpha} = \mathcal{Z}(\mathcal{N})$ ;
- (iii)  $\dim \mathcal{Z}(\mathcal{N}) = \dim \mathcal{A} = 1$ ;
- (iv) *there exists an element  $E \in \mathcal{Z}(\mathcal{N})$  such that the Lie bracket on  $\mathcal{N}$  reads*

$$[x, y] = \Omega(x, y)E \quad x, y \in \mathcal{N}.$$

*The subspaces  $\mathcal{A} \oplus \mathcal{Z}(\mathcal{N})$  and  $\mathcal{G}_\alpha$  are symplectic and orthogonal in  $(\mathcal{R}, \Omega)$ . In particular,  $\mathcal{N}$  is a Heisenberg algebra.*

*Proof.* Since  $\dim \mathcal{A} = 1$ , every root is a multiple of a given one, say  $\alpha$ . A classical lemma [8] tells us that  $\Phi \subset \{\pm\alpha, \pm 2\alpha\}$ . The hypothesis  $\dim \mathcal{G} > 3$  together with Lemma 1.1 imply (i), (ii) and (iii). For (iv), we will first prove that  $Z = \sigma E_0 + T_0 + E_0$  with  $T_0 \in \mathcal{Z}_\mathcal{K}(\mathcal{A})$  and  $E_0 \in \mathcal{Z}(\mathcal{N})$ . Indeed, let  $a \in \mathcal{A}$  and  $n, n' \in \mathcal{N}$  be such that  $Z = \sigma n + a + T_0 + n'$  according to the decomposition (3). Then  $B(Z, \mathcal{A}) = 0$  implies  $a = 0$ ; hence  $\sigma n + n' \in \mathcal{K}$ . Thus  $n = n'$  because  $\mathcal{N} \cap \mathcal{K} = 0$ . Now let  $z \in \mathcal{Z}(\mathcal{N})$ . Then  $[Z, z + \sigma z] = 0 = [T_0, z] + \sigma[T_0, z] + [\sigma n, z] + \sigma[\sigma n, z]$ . Since  $[\mathcal{G}_0, \mathcal{Z}(\mathcal{N})] \subset \mathcal{Z}(\mathcal{N})$ , one gets  $[T_0, z] = 0$ . Writing  $n = n_\alpha + n_{2\alpha}$  according to the decomposition  $\mathcal{N} = \mathcal{G}_\alpha \oplus \mathcal{G}_{2\alpha}$ , one gets  $[\sigma n_\alpha, z] + [\sigma n_{2\alpha}, z] \in \mathcal{P}$ . Hence, since  $[\sigma n_{2\alpha}, z] \in \mathcal{G}_0$ , one has  $[\sigma n_\alpha, z] \in \mathcal{G}_\alpha \cap \mathcal{P}$ . This last intersection being zero since  $[\mathcal{A}, \mathcal{P}] \subset \mathcal{K}$  implies  $\mathcal{N} \cap \mathcal{P} = 0$ . Therefore  $n = n_{2\alpha} = E_0 \in \mathcal{Z}(\mathcal{N})$  and one gets the desired form for  $Z$ .

Now, one has  $\Omega(\mathcal{A}, \mathcal{G}_\alpha) = B(Z, [\mathcal{A}, \mathcal{G}_\alpha]) = B(\sigma E_0 + E_0, \mathcal{G}_\alpha) = 0$ . Therefore  $\mathcal{G}_\alpha = (\mathcal{A} \oplus \mathcal{Z}(\mathcal{N}))^{\perp_\Omega}$  is symplectic and the table of  $\mathcal{N}$  reads  $[x, y] = \frac{1}{B(\sigma E_0, E_0)}\Omega(x, y)E_0$ .  $\blacksquare$

*Proof of Proposition 1.3.* Assume the rank to be one and  $\dim \mathcal{G} > 3$ . By distributing the terms of the Taylor series of function  $F$  (cf. Proposition 1.2), one gets :

$$\begin{aligned} & [Ad(\exp(-n))u_{\mathcal{A}} + F(ad(n))u_{\mathcal{N}}, (u \leftrightarrow v)] = \\ & [u_{\mathcal{A}} - [n, u_{\mathcal{A}}] + \mathbf{c.t.}, v_{\mathcal{N}} - \frac{1}{2}[n, v_{\mathcal{N}}]] + [u_{\mathcal{N}} - \frac{1}{2}[n, u_{\mathcal{N}}], v_{\mathcal{A}} - [n, v_{\mathcal{A}}]] + \\ & + [u_{\mathcal{N}} + \mathbf{c.t.}, v_{\mathcal{N}} + \mathbf{c.t.}] \quad (\mathbf{c.t.} = \text{central terms in } \mathcal{N}) = \\ & = [u_{\mathcal{A}}, v_{\mathcal{N}}] + [u_{\mathcal{N}}, v_{\mathcal{A}}] + [u_{\mathcal{N}}, v_{\mathcal{N}}] - \frac{1}{2}[u_{\mathcal{A}}, [n, v_{\mathcal{N}}]] - [[n, u_{\mathcal{A}}], v_{\mathcal{N}}] + [u_{\mathcal{N}}, [v_{\mathcal{A}}, n]] - \frac{1}{2}[[n, u_{\mathcal{N}}], v_{\mathcal{A}}] = \end{aligned}$$

$$\begin{aligned}
& \text{(because } [n, v_{\mathcal{N}}] \text{ and } [n, u_{\mathcal{N}}] \text{ are central in } \mathcal{N}) \\
& = [u, v] - \alpha(u_{\mathcal{A}})[n, v_{\mathcal{N}}] - [[n, u_{\mathcal{A}}], v_{\mathcal{N}}] + [u_{\mathcal{N}}, [v_{\mathcal{A}}, n]] + \alpha(v_{\mathcal{A}})[n, u_{\mathcal{N}}].
\end{aligned} \tag{8}$$

Now, observe that for  $n, N \in \mathcal{N}$  and  $A \in \mathcal{A}$ , one has

$$\begin{aligned}
[[n, A], N] &= [[n_{\alpha} + n_{2\alpha}, A], N] = [N, \alpha(A)n_{\alpha} + 2\alpha(A)n_{2\alpha}] = \\
& \alpha(A)[N, n_{\alpha}] = \alpha(A)[N, n] \quad \text{(because } \mathcal{G}_{2\alpha} = \mathcal{Z}(\mathcal{N})\text{)}.
\end{aligned}$$

Hence (8) becomes  $[u, v] - \alpha(u_{\mathcal{A}})[n, v_{\mathcal{N}}] - \alpha(u_{\mathcal{A}})[v_{\mathcal{N}}, n] + \alpha(v_{\mathcal{A}})[u_{\mathcal{N}}, n] + \alpha(v_{\mathcal{A}})[n, u_{\mathcal{N}}] = [u, v]$ . The case  $\dim \mathcal{G} = 3$  is similar and simpler.  $\blacksquare$

## 2 Guessing the product formula

Star products have been introduced in [2] as an autonomous formulation of Quantum Mechanics.

**Definition 2.1** *A star product on a symplectic manifold  $(M, \omega)$  is an associative  $\mathbb{C}$ -bilinear multiplication  $\star_{\nu}$  on the space of formal power series  $C^{\infty}(M)[[\nu]]$  such that, for all  $u, v \in C^{\infty}(M)$ , one has*

(i)

$$u \star_{\nu} v = \sum_{k=0}^{\infty} c_k(u, v) \nu^k$$

where the  $c_k$ 's are bidifferential operators on  $C^{\infty}(M)$ ;

(ii)  $c_0(u, v) = uv$ ;

(iii)  $c_1(u, v) = \frac{1}{2}\{u, v\}$  where  $\{, \}$  is the Poisson bracket associated to the symplectic form  $\omega$ ;

(iv)  $u \star_{\nu} 1 = 1 \star_{\nu} u = u$ .

**Example 2.1** Let  $(V, \Omega)$  be a symplectic vector space of dimension  $2n$  and write  $\Omega(x, y) = \langle x, Jy \rangle$  with  $J \in so(V)$ . Then evaluating Rieffel's product (1) of two compactly supported functions  $u, v \in C_c^{\infty}(V)$  at a point  $x \in V$ , one re-finds the old expression of the **Weyl product** :

$$(u \star_{\hbar}^W v)(x) \stackrel{\text{def.}}{=} (u \star_{\hbar J} v)(x) = \hbar^{-2n} \int_{V \times V} e^{\frac{2i}{\hbar} S^0(x, y, z)} u(x) v(y) dy dz \tag{9}$$

with

$$S^0(x, y, z) = \langle x, Jy \rangle + \langle y, Jz \rangle + \langle z, Jx \rangle,$$

as explained in the introduction. Recall that the Schwartz space  $\mathcal{S}(V)$  is stable under Weyl's product [7]. Performing a stationary phase method on the oscillatory integral (9), one gets an asymptotic expansion in the parameter  $\nu = \frac{\hbar}{2i}$  :

$$u \star_{\hbar J} v \sim uv + \nu \{u, v\} + \sum_{k=2}^{\infty} \frac{\nu^k}{k!} \sum_{\substack{i_1 \dots i_k \\ j_1 \dots j_k}} \Omega^{i_1 j_1} \dots \Omega^{i_k j_k} \partial_{i_1 \dots i_k} u \cdot \partial_{j_1 \dots j_k} v. \tag{10}$$

The expression in the RHS of (10) actually defines a star product on  $C^{\infty}(M)[[\nu]]$  called the **Moyal star product**. It will be denoted by  $\star_{\nu}^M$ .

## 2.1 $AN$ -covariant Moyal star products

In this section, we adapt to our situation old techniques from star representation theory [1, 2, 5]. First, we recall the notion of covariant star product.

**Definition 2.2** *let  $(M, \omega)$  be a symplectic manifold on which a connected Lie group  $G$  acts in a strongly Hamiltonian way. Let  $\mathcal{G}$  be the Lie algebra of  $G$  and denote by*

$$\mathcal{G} \xrightarrow{\lambda} C^\infty(M) : X \rightarrow \lambda_X$$

*the (dual) moment map i.e.*

$$i_{X^*}\omega = -d\lambda_X.$$

*A star product  $\star_\nu$  on  $C^\infty(M)[[\nu]]$  is said to be  $G$ -covariant if*

$$[\lambda_X, \lambda_Y]_{\star_\nu} \stackrel{\text{def.}}{=} \lambda_X \star_\nu \lambda_Y - \lambda_Y \star_\nu \lambda_X = 2\nu\{\lambda_X, \lambda_Y\}.$$

**Proposition 2.1** *Within the assumptions and notations of Proposition 1.2, let  $R \times \mathcal{R} \xrightarrow{\tau} \mathcal{R}$  be the action defined by*

$$\tau_g(r) = \mathcal{I}^{-1}g\mathcal{I}(r).$$

*Assume  $\dim \mathcal{R} \geq 4$  (i.e.  $\dim \mathcal{G} > 3$ ). Then, this action is Hamiltonian with respect to the constant symplectic structure  $\Omega$  on  $\mathcal{R}$ . Moreover, the Hamiltonian functions associated to the infinitesimal action are*

$$\begin{aligned} \lambda_A(r) &= 2\alpha(A)B(\sigma E, E)n_E \quad (A \in \mathcal{A}), \\ \lambda_y(r) &= e^{-\alpha(a)}\Omega(n, y) \quad (y \in \mathcal{G}_\alpha), \\ \lambda_E(r) &= e^{-2\alpha(a)}B(\sigma E, E), \end{aligned}$$

*where  $r = (a, n)$  and  $n = n_\alpha + n_E E$  according to the decomposition  $\mathcal{N} = \mathcal{G}_\alpha \oplus \mathbb{R}E$  (cf. Lemma 1.2). In particular, every such Hamiltonian is linear in  $\mathcal{N}$ , therefore the Moyal star product on  $(\mathcal{R}, \Omega)$  is  $R$ -covariant.*

*Proof.*

$$\begin{aligned} \lambda_A(r) &= B(\text{Ad}(\exp a \exp n)Z_0, A) \\ &= B(Z_0, \text{Ad}(\exp -n)A) \\ &= B(Z_0, A - [n, A] + \frac{1}{2}[n, [n, A]]) \\ &= B(\sigma E, A - [n, A] + \frac{1}{2}[n, [n, A]]) \\ &= B(\sigma E, 2\alpha(A)n_E E + \frac{1}{2}[\alpha(A)n_\alpha, n_\alpha]) \\ &= 2\alpha(A)B(\sigma E, E)n_E; \\ \lambda_y(r) &= B(\text{Ad}(\exp a \exp -n)Z_0, y) \\ &= e^{-\alpha(a)}B(Z_0, \text{Ad}(\exp -n)y) \\ &= e^{-\alpha(a)}B(Z_0, y - [n, y]) \\ &= e^{-\alpha(a)}B(Z_0, -[n, y]) = e^{-\alpha(a)}\Omega(n, y); \\ \lambda_E(r) &= B(\text{Ad}(\exp a \exp n)Z_0, E) \\ &= e^{-2\alpha(a)}B(Z_0, \text{Ad}(\exp -n)E) \\ &= e^{-2\alpha(a)}B(Z_0, E). \end{aligned}$$

■



When covariant, a star product yields a representation of  $\mathcal{G}$  on  $C^\infty(M)[[\nu]]$  :

$$\mathcal{G} \xrightarrow{\rho_\nu} \text{End}(C^\infty(M)[[\nu]])$$

$$\rho_\nu(X)u = \frac{1}{2\nu}[\lambda_X, u]_{\star_\nu}.$$

In order to compute the representation  $\rho_\nu$  in our context, we observe

**Lemma 2.1** *let  $(\mathcal{R}, \Omega)$  be a symplectic vector space. Let  $\mathcal{U}$  be a codimension 2 symplectic subspace of  $\mathcal{R}$  and let  $A$  and  $E$  be generators of  $\mathcal{U}^\perp$ . Then, for every linear form  $\mu \in \mathcal{U}^*$  and every smooth function  $\epsilon \in C^\infty(\mathbb{R}A)$ , one has*

$$\Omega^{i_1 j_1} \dots \Omega^{i_k j_k} \partial_{i_1 \dots i_k}(\epsilon \otimes \mu) \partial_{j_1 \dots j_k}(u) = k \partial_A^{k-1} \epsilon \partial_{\sharp \mu} \partial_E^{k-1} u + \mu \partial_A^k \epsilon \partial_E^k u \quad (u \in C^\infty(\mathcal{R})),$$

where  $\sharp \mu$  is defined by  $\Omega(\sharp \mu, \cdot) = -\mu$ .

*Proof.* One has  $\partial_E(\epsilon \otimes \mu) = 0$  and  $\partial_{\mathcal{U}}^\ell(\epsilon \otimes \mu) = 0$  as soon as  $\ell \geq 2$ . Hence  $\partial_{i_1 \dots i_k}(\epsilon \otimes \mu) \neq 0$  only if the  $k$ -tuple  $(i_1 \dots i_k)$  contains either one or zero element of  $\mathcal{U}$ ; all the other ones being  $A$ 's. There are  $k$  such  $k$ -tuples for a given element of  $\mathcal{U}$ . Therefore, the only  $(j_1 \dots j_k)$ 's yielding non zero contributions in the LHS contain either one or zero element of  $\mathcal{U}$  (conjugated with the one in the corresponding  $(i_1 \dots i_k)$ ) and  $E$ 's. Therefore, one gets

$$LHS = k \Omega_{\mathcal{U}}^{\alpha \beta} \partial_A^{k-1} \partial_\alpha(\epsilon \otimes \mu) \partial_\beta \partial_E^{k-1} u + \partial_A^k(\epsilon \otimes \mu) \partial_E^k u.$$

One concludes using  $\Omega^{\alpha \beta} \partial_\alpha \mu \partial_\beta = \partial_{\sharp \mu}$ . ■

This implies that for  $y \in \mathcal{G}_\alpha$ , one has

$$\begin{aligned} \frac{1}{2\nu}[\lambda_y, u]_{\star_\nu^M} &= \frac{2}{2\nu} \sum_k \frac{\nu^{2k+1}}{(2k+1)!} \left\{ (2k+1)(-\alpha(A))^{2k} e^{-\alpha(A)} \partial_y \partial_E^{2k} u + \Omega(n, y)(-\alpha(A))^{2k+1} e^{-\alpha(A)} \partial_E^{2k+1} u \right\} \\ &= e^{-\alpha(A)} \cosh(\nu(-\alpha(A))\partial_E) \partial_y u + \frac{1}{\nu} \Omega(n, y) \sinh(\nu(-\alpha(A))\partial_E) e^{-\alpha(A)} u. \end{aligned}$$

Also

$$\frac{1}{2\nu}[\lambda_A, u]_{\star_\nu^M} = \partial_A u$$

and

$$\begin{aligned} \frac{1}{2\nu}[\lambda_E, u]_{\star_\nu^M} &= \frac{2}{2\nu} \sum_k \frac{\nu^{2k+1}}{(2k+1)!} B(\sigma E, E)(-2\alpha(A))^{2k+1} e^{-2\alpha(A)} \partial_E^{2k+1} u \\ &= \frac{1}{\nu} B(\sigma E, E) \sinh(\nu(-2\alpha(A))\partial_E) e^{-2\alpha(A)} u. \end{aligned}$$

Regarding these expressions, it is tempting to take the partial Fourier transform in the  $E$ -variable in order to obtain a so called “multiplicative representation”.

Writing an element  $r \in \mathcal{R}$  as

$$r = aA + x + zE \text{ with } x \in \mathcal{G}_\alpha,$$

we set, for (reasonable)  $u \in C^\infty(\mathcal{R})$ ,

$$F(u)(a, x, \xi) = \hat{u}(a, x, \xi) = \int_{\mathcal{Z}(\mathcal{N})} e^{-i\xi z} u(aA + x + zE) dz.$$

One then has  $F(\partial_E u) = i\xi \hat{u}$  which yields

$$F(\rho_\nu(y)u) = e^{-\alpha(A)} \cosh(\nu(-\alpha(A))i\xi) \partial_y \hat{u} + \frac{1}{\nu} \Omega(x, y) \sinh(\nu(-\alpha(A))i\xi) e^{-\alpha(A)} \hat{u};$$

$$F(\rho_\nu(A)u) = \partial_A \hat{u};$$

$$F(\rho_\nu(E)u) = \frac{1}{\nu} B(\sigma E, E) \sinh(\nu(-2\alpha(A))i\xi) e^{-2\alpha(A)} \hat{u}.$$

Choosing  $A$  such that  $\alpha(A) = 1$  and setting  $\nu = \frac{\hbar}{2i}$ , one gets

$$\begin{aligned}\hat{\rho}_{\hbar}(y)\hat{u} &= e^{-a} \cosh\left(\frac{\hbar}{2}\xi\right) \partial_y \hat{u} + \frac{2i}{\hbar} \Omega(x, y) \sinh\left(\frac{\hbar}{2}\xi\right) e^{-a} \hat{u}; \\ \hat{\rho}_{\hbar}(A)\hat{u} &= \partial_A \hat{u}; \\ \hat{\rho}_{\hbar}(E)\hat{u} &= -\frac{2i}{\hbar} B(\sigma E, E) \sinh(\hbar\xi) e^{-2a} \hat{u},\end{aligned}$$

where  $\hat{\rho}_{\hbar}$  is the representation of  $\mathcal{G}$  defined by

$$\hat{\rho}_{\hbar}(X)\hat{u} = F(\rho_{\frac{\hbar}{2i}}u).$$

This is a multiplicative representation. We now change the coordinates following

$$(a, x, \xi) = (a', \cosh(\frac{\hbar}{2}\xi')x', \xi') \stackrel{\text{def.}}{=} \varphi_{\hbar}(a', x', \xi'). \quad (11)$$

This yields

$$\hat{\rho}_{\hbar}(y)f(a', x', \xi') = e^{-a'} \partial_{y'} f - \Omega(x', y) e^{-a'} \frac{i}{\hbar} \sinh(\hbar\xi') f;$$

the rest being unchanged.

**Definition 2.3** We denote by  $c_{\hbar} \in \Omega^1(\mathcal{R})$  the smooth one-form on  $\mathcal{R}$  defined by

$$(c_{\hbar})_{(a, x, \xi)}(X) = -e^{-a} \frac{\sinh(\hbar\xi)}{\hbar} (\Omega(x, X) + 2e^{-a} B(Z_0, X)) \quad X \in T_{(a, x, \xi)}(\mathcal{R}).$$

One then gets

**Lemma 2.2** Under the transformation  $\varphi_{\hbar}^* \circ F$ , the star representation of  $\mathcal{R} = \text{Lie}(\mathcal{R})$  on  $C^\infty(\mathcal{R})[[\hbar]]$  is multiplicative and reads as

$$\pi_{\hbar}(X)f(a, x, \xi) \stackrel{\text{def.}}{=} (e^{-a} X_{\alpha} + X_{\mathcal{A}}) \cdot f + i c_{\hbar}(X)f \quad (f \in C^\infty(\mathcal{R}))$$

where  $X = X_{\mathcal{A}} + X_{\alpha} + X_E$  according to the decomposition  $\mathcal{R} = \mathcal{A} \oplus \mathcal{G}_{\alpha} \oplus \mathbb{R}E$  and where

$$\pi_{\hbar}(X) = \varphi_{\hbar}^* \circ \hat{\rho}_{\hbar}(X) \circ (\varphi_{\hbar}^{-1})^* = \varphi_{\hbar}^* \circ F \circ \rho_{\frac{\hbar}{2i}}(X) \circ F^{-1} \circ (\varphi_{\hbar}^{-1})^*.$$

## 2.2 The $\mathcal{Z}_{\hbar}$ -transform

**Definition 2.4** For  $u \in C^\infty(\mathcal{R})$  integrable, we define the  $\mathcal{Z}_{\hbar}$ -transform by

$$(\mathcal{Z}_{\hbar}(u))(a, x, \xi) = \int e^{-\frac{i}{\hbar} \sinh(\hbar\xi)z} u(a, x, z) dz.$$

The formal (commutative) product obtained by transporting the pointwise multiplication of functions via  $\mathcal{Z}_{\hbar}$  is denoted by  $\bullet_{\hbar}$  :

$$f \bullet_{\hbar} g \stackrel{\text{def.}}{=} \mathcal{Z}_{\hbar}(\mathcal{Z}_{\hbar}^{-1} f \cdot \mathcal{Z}_{\hbar}^{-1} g)$$

(whenever this expression makes sense).

**Theorem 2.1** Under representation  $\pi_{\hbar}$ , the Lie algebra  $\mathcal{R}$  acts by derivations with respect to the commutative product  $\bullet_{\hbar}$  i.e. one has formally

$$\pi_{\hbar}(f \bullet_{\hbar} g) = (\pi_{\hbar}(X)f) \bullet_{\hbar} g + f \bullet_{\hbar} (\pi_{\hbar}(X)g).$$

*Proof.* It is sufficient to prove that, for all  $X \in \mathcal{R}$ ,

$$X^{\hbar} \stackrel{\text{def.}}{=} \mathcal{Z}_{\hbar}^{-1} \circ m_{c_{\hbar}(X)} \circ \mathcal{Z}_{\hbar} \text{ is a vector field on } \mathcal{R} \quad (12)$$

( $m_{c_{\hbar}(X)}$  denotes the multiplication by  $c_{\hbar}(X) : m_{c_{\hbar}(X)}(f) = c_{\hbar}(X)f$ ). Indeed, if (12) holds one has

$$\begin{aligned} c_{\hbar}(X)f \bullet_{\hbar} g + f \bullet_{\hbar} c_{\hbar}(X)g &= \\ \mathcal{Z}_{\hbar}(X^{\hbar} \mathcal{Z}_{\hbar}^{-1} f \cdot \mathcal{Z}_{\hbar}^{-1} g + \mathcal{Z}_{\hbar}^{-1} f \cdot X^{\hbar} \mathcal{Z}_{\hbar}^{-1} g) &= \\ \mathcal{Z}_{\hbar} X^{\hbar} (\mathcal{Z}_{\hbar}^{-1} f \cdot \mathcal{Z}_{\hbar}^{-1} g) &= \\ c_{\hbar}(X)(f \bullet_{\hbar} g). \end{aligned}$$

Therefore, since the vector part  $\tilde{X} = e^{-a}X_{\alpha} + X_{\mathcal{A}}$  of  $\pi_{\hbar}(X)$  does not involve the  $E$ -variable, its action commutes with the  $\mathcal{Z}_{\hbar}$ -transform and one gets

$$\begin{aligned} \pi_{\hbar}(f \bullet_{\hbar} g) &= \\ \mathcal{Z}_{\hbar}(\tilde{X}(\mathcal{Z}_{\hbar}^{-1} f \cdot \mathcal{Z}_{\hbar}^{-1} g)) + i c_{\hbar}(X)(f \bullet_{\hbar} g) &= \\ \mathcal{Z}_{\hbar}((\mathcal{Z}_{\hbar}^{-1} \tilde{X} f) \mathcal{Z}_{\hbar}^{-1} g + \mathcal{Z}_{\hbar}^{-1} f (\mathcal{Z}_{\hbar}^{-1} \tilde{X} g)) + i(c_{\hbar}(X)f) \bullet_{\hbar} g + i f \bullet_{\hbar} (c_{\hbar}(X)f) &= \\ (\tilde{X} f) \bullet_{\hbar} g + f \bullet_{\hbar} (\tilde{X} g) + i(c_{\hbar}(X)f) \bullet_{\hbar} g + i f \bullet_{\hbar} (c_{\hbar}(X)f) &= \\ (\pi_{\hbar}(X)f) \bullet_{\hbar} g + f \bullet_{\hbar} (\pi_{\hbar}(X)g). \end{aligned}$$

We now prove assertion (12). For  $y \in \mathcal{G}_{\alpha}$ , one has

$$c_{\hbar}(y) \mathcal{Z}_{\hbar}(u) = e^{-a} \Omega(y, x) \frac{\sinh(\hbar \xi)}{\hbar} \int e^{-\frac{i}{\hbar} \sinh(\hbar \xi) z} u(a, x, z) dz = -i e^{-a} \Omega(y, x) \mathcal{Z}_{\hbar}(\partial_z u).$$

Hence, since the  $\mathcal{Z}_{\hbar}$ -transform only involves the  $E$ -variable,

$$\mathcal{Z}_{\hbar}^{-1} \circ m_{c_{\hbar}(y)} \circ \mathcal{Z}_{\hbar} = -i e^{-a} \Omega(y, x) \partial_z.$$

For  $A \in \mathcal{A}$ , one has  $c_{\hbar}(A) = 0$ . For  $E \in \mathcal{Z}(\mathcal{N})$ , one has

$$\mathcal{Z}_{\hbar}(\partial_E u) = \int e^{-\frac{i}{\hbar} \sinh(\hbar \xi) z} \partial_E u dz = \frac{i}{\hbar} \sinh(\hbar \xi) \mathcal{Z}_{\hbar}(u) = -i \frac{e^{2a} c_{\hbar}(E)}{2B(Z_0, E)} \mathcal{Z}_{\hbar}(u),$$

hence

$$\mathcal{Z}_{\hbar}^{-1} c_{\hbar}(E) \mathcal{Z}_{\hbar}(u) = 2i B(Z_0, E) e^{-2a} \partial_E u. \quad \blacksquare$$

If one interprets the commutative product  $\bullet_{\hbar}$  as the underlying product to the algebra of functions on a commutative  $\hbar$ -dependent manifold, say  $M_{\hbar}$ , its invariance under  $\hat{\rho}$  tells us that  $\mathcal{G}$  is realized via  $\hat{\rho}$  as a subalgebra of tangent vector fields over  $M_{\hbar}$ .

At this level, we want

- (a) to identify the infinitesimal action  $\mathcal{G} \rightarrow \Gamma(TM_{\hbar})$ ;
- (b) to identify the product on  $C^{\infty}(\mathcal{R})[[\hbar]]$  defined by

$$u \star_{\hbar} v = T_{\hbar}^{-1} (T_{\hbar} u \star_{\frac{M}{2i}} T_{\hbar} v)$$

$$\text{with } T_{\hbar} = F^{-1} \circ (\varphi^{-1})^* \circ \mathcal{Z}_{\hbar}.$$

Formally  $\star_{\hbar}$  is indeed a quantization of  $(\mathcal{R}, \Omega)$  since  $\lim_{\hbar \rightarrow 0} T_{\hbar} = id$ . The following proposition answers question (a).

**Proposition 2.2** *For all  $X \in \mathcal{R}$ , one has*

$$\mathcal{Z}_{\hbar}^{-1} \circ \pi_{\hbar}(X) \circ \mathcal{Z}_{\hbar} = X^*,$$

*whenever this expression makes sense.*

*Proof.* First, one has

$$\mathcal{Z}_h^{-1} \pi_h(A) \mathcal{Z}_h(u) = \mathcal{Z}_h^{-1} \partial_A \mathcal{Z}_h(u) = \partial_A u = A^* u.$$

Second, for  $X = y + xE \in \mathcal{N}$  ( $y \in \mathcal{G}_\alpha$ ), one has

$$X^* u = \{\lambda_X, u\} = e^{-a} \partial_y u - \Omega(x, y) e^{-a} \partial_E u - 2B(Z_0, Z) e^{-2a} \partial_E u$$

(cf. Proposition 2.1). Hence

$$\begin{aligned} \mathcal{Z}_h X^* u &= \\ \int e^{-\frac{i}{\hbar} \sinh(\hbar \xi) z} \{ e^{-a} \partial_y u - \Omega(x, y) e^{-a} \partial_E u - 2B(Z_0, Z) e^{-2a} \partial_E u \} &= \\ -\frac{i}{\hbar} \sinh(\hbar \xi) e^{-a} (\Omega(x, y) + 2B(Z_0, Z) e^{-a}) \mathcal{Z}_h(u) + e^{-a} \partial_y \mathcal{Z}_h(u) &= \\ i c_h(X) \mathcal{Z}_h(u) + e^{-a} \partial_y \mathcal{Z}_h(u) &= \\ \pi_h(X) \mathcal{Z}_h(u). \end{aligned}$$

■

Therefore  $M_h$  can be  $R$ -equivariantly identified with  $\mathcal{R}$ , which implies that the star product  $\star_h$  on  $(\mathcal{R}, \Omega)$  described in (b) is  $R$ -invariant.

In order to define function algebras which will be stable under the product  $\star_h$ , we will transport the structure of the Schwartz space— which is stable under the Weyl product (9)— via the “equivalence”  $T_h$ .

### 3 Strict Quantization

In this section, we adopt the following notation. If  $V$  is a finite dimensional vector space, we denote by  $\mathcal{S}(V)$  (resp.  $\mathcal{S}'(V)$ ) the space of Schwartz functions (resp. tempered distributions) on  $V$ .

**Lemma 3.1** *Let  $\phi_h : \mathcal{R} \rightarrow \mathcal{R}$  be the diffeomorphism defined by*

$$\phi_h(a, x, \xi) = (a, \frac{1}{\cosh(\frac{\hbar}{2}\xi)} x, \frac{1}{\hbar} \sinh(\hbar \xi)).$$

*Then, one has*

- (i)  $\phi_h^* \mathcal{S}(\mathcal{R}) \subset \mathcal{S}(\mathcal{R})$ ,
- (ii)  $(\phi_h^{-1})^* \mathcal{S}(\mathcal{R}) \subset \mathcal{S}'(\mathcal{R})$ .

*Proof.* For the sake of simplicity, we will only prove that, if  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined by  $\phi(x, y) = (\text{sech}(\frac{y}{2})x, \sinh(y))$ , then  $\phi^* \mathcal{S}(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2)$  and  $(\phi^{-1})^* \mathcal{S}(\mathbb{R}^2) \subset \mathcal{S}'(\mathbb{R}^2)$ . The proofs of items (i) and (ii) being entirely similar.

First, one has

$$\begin{aligned} \phi^{-1}(x, y) &= (\frac{\sqrt{2}}{2} (1 + \sqrt{1 + y^2})^{\frac{1}{2}} x, \text{arcsinh}(y)), \\ \phi_{\star(x, y)} &= \begin{pmatrix} \text{sech}(\frac{y}{2}) & -\frac{x}{2} \tanh(\frac{y}{2}) \text{sech}(\frac{y}{2}) \\ 0 & \cosh(y) \end{pmatrix} \text{ and} \\ \frac{\sqrt{2}}{2} (1 + \sqrt{1 + y^2})^{\frac{1}{2}} &= \cosh(\frac{\text{arcsinh}(y)}{2}). \end{aligned} \tag{13}$$

Therefore, setting  $p_{n, m}(x, y) = x^n y^m$ ,  $p_{n, m} \circ \phi^{-1}$  has still polynomial growth. This implies that for all  $n, m$ ,  $\sup_a \{|p_{n, m}(a) \phi^* u(a)|\} < \sup_a \{|p_{N, M}(a) u(a)|\}$  for some  $N, M$ . This last expression being finite if  $u \in \mathcal{S}(\mathbb{R}^2)$ . For derivatives of  $\phi^*(u)$ , one needs to control the asymptotic behavior of  $\phi_{\star \phi^{-1}(x, y)}$ . Formulas (13) imply that  $\|\phi_{\star \phi^{-1}(x, y)}\|$  has polynomial growth. This shows that  $\sup\{|p_{n, m} D \phi^* u|\} < \infty$ . An induction argument

then yields  $\phi^* \mathcal{S}(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2)$ .

Now, one wants to find  $N, M \geq 0$  such that

$$\int_U |x^{-N} y^{-M} (\phi^{-1})^* u(x, y)| dx dy$$

is finite ( $U$  is the complementary subset of some compact neighborhood of the origin). Changing the variables following  $a \rightarrow \phi(a)$ , this integral becomes

$$\begin{aligned} & \int_{U'} \left| \frac{1}{p_{N,M}(\phi(a))} u(a) \right| |\text{Jac}_\phi(a)| da = \\ & \int_{U'} \frac{|2 \sinh(\frac{y}{2})|}{\left| \left( \frac{x}{\cosh(\frac{y}{2})} \right)^N (\sinh(y))^M \right|} |u(a)| da = \\ & \int_{U'} \frac{2^{1-M} |\sinh(\frac{y}{2})|^{1-M}}{x^N \cosh(\frac{y}{2})^{N-M}} |u(a)| da \end{aligned}$$

which is finite as soon as  $N \geq M \geq 1$  ( $U'$  is of the same type as  $U$ ). For derivatives, one needs to control  $\|\phi_{\star\phi(a)}^{-1}(A)\|$  i.e. the norm of the inverse matrix  $[\phi_\star]^{-1}$ , which, by looking at formulas (13), has polynomial growth. An induction argument then yields  $(\phi^{-1})^* \mathcal{S}(\mathbb{R}^2) \subset \mathcal{S}'(\mathbb{R}^2)$ . ■

Lemma 3.1 allows us to define the following linear injection :

$$\begin{aligned} \tau_{\hbar} &: \mathcal{S}(\mathcal{R}) \rightarrow \mathcal{S}'(\mathcal{R}) \\ \tau_{\hbar} &\stackrel{\text{def.}}{=} F^{-1} \circ (\phi_{\hbar}^{-1})^* \circ F \end{aligned}$$

where one extends the Fourier transform to the tempered distributions. We then set

$$\mathcal{E}_{\hbar} \stackrel{\text{def.}}{=} \tau_{\hbar} \mathcal{S}(\mathcal{R}) \subset \mathcal{S}'(\mathcal{R}).$$

**Lemma 3.2** (i)  $\mathcal{S}(\mathcal{R}) \subset \mathcal{E}_{\hbar}$ .

(ii) The map  $T_{\hbar} : \mathcal{S}(\mathcal{R}) \rightarrow \mathcal{S}(\mathcal{R}) : T_{\hbar} = F^{-1} \circ \phi_{\hbar}^* \circ F$  extends to  $\mathcal{E}_{\hbar}$  as a linear isomorphism  $T_{\hbar} : \mathcal{E}_{\hbar} \rightarrow \mathcal{S}(\mathcal{R})$ .

(iii) One has  $T_{\hbar} \circ \tau_{\hbar} = \text{id}_{\mathcal{S}(\mathcal{R})}$  and  $\tau_{\hbar} \circ T_{\hbar}|_{\mathcal{S}(\mathcal{R})} = \text{id}_{\mathcal{S}(\mathcal{R})}$ .

*Proof.* For  $u \in \mathcal{S}(\mathcal{R})$ , one has  $T_{\hbar}(u) \in \mathcal{S}(\mathcal{R})$  hence  $\tau_{\hbar} T_{\hbar}(u) = u \in \mathcal{E}_{\hbar}$ . The rest is obvious. ■

**Theorem 3.1** (i) For  $a, b \in \mathcal{E}_{\hbar}$ , the formula

$$a \star_{\hbar} b \stackrel{\text{def.}}{=} \tau_{\hbar}(T_{\hbar} a \star_{\hbar}^W T_{\hbar} b)$$

defines an associative algebra structure on  $\mathcal{E}_{\hbar}$  ( $\star_{\hbar}^W$  denotes the Weyl product on  $\mathcal{S}(\mathcal{R})$ , see Formula (9)).

(ii) For  $u, v \in \mathcal{S}(\mathcal{R}) \subset \mathcal{E}_{\hbar}$ , the product  $\star_{\hbar}$  reads

$$(u \star_{\hbar} v)(a_0, x_0, z_0) =$$

$$\begin{aligned} & \frac{1}{\hbar^{2 \dim \mathcal{R}}} \int_{\mathcal{R} \times \mathcal{R}} \cosh(2(a_1 - a_2)) \cosh(a_2 - a_0) \cosh(a_0 - a_1) \times \\ & \times \exp \left( \frac{2i}{\hbar} \left\{ S^0(\cosh(a_1 - a_2)x_0, \cosh(a_2 - a_0)x_1, \cosh(a_0 - a_1)x_2) - \frac{1}{2} \oint_{0,1,2} \sinh(2(a_0 - a_1))z_2 \right\} \right) \times \\ & \times u(a_1, x_1, z_1) v(a_2, x_2, z_2) da_1 da_2 dx_1 dx_2 dz_1 dz_2 \end{aligned} \quad (14)$$

where  $S^0$  is the phase for the Weyl product (cf. Formula (9)) and where  $\oint_{0,1,2}$  stands for cyclic summation.

(iii) In the Iwasawa coordinates  $\mathcal{R} \xrightarrow{\mathcal{I}} R$ , the group multiplication law reads

$$L_{(a,x,z)}(a', x', z') = \left( a + a', e^{-a'} x + x', e^{-2a'} z + z' + \frac{1}{2}\Omega(x, x')e^{-a'} \right).$$

Both phase and amplitude occurring in formula (14) are invariant under the left action  $L : R \times R \rightarrow R$ .

*Proof.* We perform the computation which leads to formula (14). On the one hand, we have

$$(T_{\hbar} u \star_{\hbar}^W T_{\hbar} v)(a_0, x_0, z_0) = \int (T_{\hbar} u)(a_1, x_1, z_1) (T_{\hbar} v)(a_2, x_2, z_2) \exp \left( \frac{2i}{\hbar} S^0(p_0, p_1, p_2) \right) dp_1 dp_2$$

with  $p_i = (a_i, x_i, z_i)$ , that is

$$\begin{aligned} & \int e^{i\Omega(\xi_1, z_1)} (\phi_{\hbar}^* \hat{u})(a_1, x_1, \xi_1) e^{i\Omega(\xi_2, z_2)} (\phi_{\hbar}^* \hat{v})(a_2, x_2, \xi_2) \times \\ & \quad \times \exp \left\{ \frac{2i}{\hbar} (\Omega(a_0, z_1) - \Omega(a_1, z_0) + \Omega(x_0, x_1) + \right. \\ & \quad \left. + \Omega(a_1, z_2) - \Omega(a_2, z_1) + \Omega(x_1, x_2) + \Omega(a_2, z_0) - \Omega(a_0, z_2) + \Omega(x_2, x_0)) \right\} \\ & \quad (\text{we omit the } dp_i \text{'s and other such differentials}) \\ & = \int \exp \left( i \left\{ \Omega(\xi_1 + \frac{2}{\hbar}(a_0 - a_2), z_1) + \Omega(\xi_2 + \frac{2}{\hbar}(a_1 - a_0), z_2) \right\} + \frac{2i}{\hbar} \Omega(a_2 - a_1, z_0) + \frac{2i}{\hbar} S^0(x_0, x_1, x_2) \right) \times \\ & \quad \times (\phi_{\hbar}^* \hat{u})(a_1, x_1, \xi_1) (\phi_{\hbar}^* \hat{v})(a_2, x_2, \xi_2) \\ & = \int \exp \left( \frac{2i}{\hbar} \left\{ S^0(x_0, x_1, x_2) + \Omega(a_2 - a_1, z_0) \right\} \right) (\phi_{\hbar}^* \hat{u})(a_1, x_1, \frac{2}{\hbar}(a_2 - a_0)) (\phi_{\hbar}^* \hat{v})(a_2, x_2, \frac{2}{\hbar}(a_0 - a_1)). \end{aligned}$$

On the second hand, one has

$$\begin{aligned} \tau_{\hbar} u(a_0, x_0, z_0) &= \int e^{i\Omega(\xi, z_0)} (\phi_{\hbar}^{-1})^* \hat{u}(a_0, x_0, \xi) d\xi \\ &= \int e^{i\Omega(\xi, z_0)} \hat{u} \left( a_0, \cosh\left(\frac{1}{2}\operatorname{arcsinh}(\hbar\xi)\right)x_0, \frac{1}{\hbar}\operatorname{arcsinh}(\hbar\xi) \right) \\ &= \int \exp \left( i \left\{ \Omega(\xi, z_0) - \Omega\left(\frac{1}{\hbar}\operatorname{arcsinh}(\hbar\xi), z\right) \right\} \right) u \left( a_0, \cosh\left(\frac{1}{2}\operatorname{arcsinh}(\hbar\xi)\right)x_0, z \right) dz d\xi. \end{aligned}$$

Therefore, one gets

$$\begin{aligned} \tau_{\hbar}(T_{\hbar} u \star_{\hbar}^W T_{\hbar} v)(a_0, x_0, z_0) &= \int \exp \left( \frac{2i}{\hbar} \left\{ S^0(\cosh(\frac{1}{2}\operatorname{arcsinh}(\hbar\xi))x_0, x_1, x_2) + \Omega(a_2 - a_1, z) \right\} \right) \times \\ & \quad \times (\phi_{\hbar}^* \hat{u})(a_1, x_1, \frac{2}{\hbar}(a_2 - a_0)) (\phi_{\hbar}^* \hat{v})(a_2, x_2, \frac{2}{\hbar}(a_0 - a_1)) \exp \left( i \left\{ \Omega(\xi, z_0) - \Omega\left(\frac{1}{\hbar}\operatorname{arcsinh}(\hbar\xi), z\right) \right\} \right) \end{aligned}$$

which is, changing the variables following  $\eta = \operatorname{arcsinh}(\hbar\xi)$  :

$$\begin{aligned} & \int \exp \left( \frac{2i}{\hbar} \left\{ S^0(\cosh(\frac{1}{2}\eta)x_0, x_1, x_2) + \Omega(a_2 - a_1, z) \right\} \right) (\phi_{\hbar}^* \hat{u})(a_1, x_1, \frac{2}{\hbar}(a_2 - a_0)) (\phi_{\hbar}^* \hat{v})(a_2, x_2, \frac{2}{\hbar}(a_0 - a_1)) \times \\ & \quad \times \exp \left( i \left\{ \Omega\left(\frac{1}{\hbar}\sinh(\eta), z_0\right) - \Omega\left(\frac{1}{\hbar}\eta, z\right) \right\} \right) \frac{1}{\hbar} \cosh(\eta) \end{aligned}$$

$$\begin{aligned}
&= \int \exp \left( \frac{2i}{\hbar} \left\{ S^0(\cosh(\frac{1}{2}\eta)x_0, x_1, x_2) + \Omega(a_2 - a_1, z) \right\} \right) \times \\
&\times \exp \left( -i \left\{ \Omega(\frac{1}{\hbar} \sinh(2(a_2 - a_0)), z_1) + \Omega(\frac{1}{\hbar} \sinh(2(a_0 - a_1)), z_2) \right\} \right) \times \\
&\times u(a_1, \cosh(a_2 - a_0)^{-1}x_1, z_1) v(a_2, \cosh(a_0 - a_1)^{-1}x_2, z_2) \exp \left( i \left\{ \Omega(\frac{1}{\hbar} \sinh(\eta), z_0) - \Omega(\frac{1}{\hbar} \eta, z) \right\} \right) \frac{1}{\hbar} \cosh(\eta)
\end{aligned}$$

which yields, changing the variables following  $\eta \leftarrow \frac{1}{\hbar}\eta$  :

$$\begin{aligned}
&= \int \exp \left( \frac{2i}{\hbar} S^0(\cosh(\frac{\hbar}{2}\eta)x_0, x_1, x_2) \right) u(a_1, \cosh(a_2 - a_0)^{-1}x_1, z_1) v(a_2, \cosh(a_0 - a_1)^{-1}x_2, z_2) \times \\
&\times \exp \left( i \left\{ -\Omega(\frac{1}{\hbar} \sinh(2(a_2 - a_0)), z_1) - \Omega(\frac{1}{\hbar} \sinh(2(a_0 - a_1)), z_2) + \Omega(\frac{1}{\hbar} \sinh(\hbar\eta), z_0) \right\} \right) \times \\
&\times \cosh(\hbar\eta) \exp(i\Omega(\frac{2}{\hbar}(a_2 - a_1) - \eta, z)) \\
&= \int \exp \left( \frac{2i}{\hbar} S^0(\cosh(a_1 - a_2)x_0, x_1, x_2) \right) u(a_1, \cosh(a_2 - a_0)^{-1}x_1, z_1) v(a_2, \cosh(a_0 - a_1)^{-1}x_2, z_2) \times \\
&\times \exp \left( \frac{-i}{\hbar} \left\{ \Omega(\sinh(2(a_2 - a_0)), z_1) + \Omega(\sinh(2(a_0 - a_1)), z_2) + \Omega(\sinh(2(a_1 - a_2)), z_0) \right\} \right) \cosh(2(a_1 - a_2))
\end{aligned}$$

which, after changing the variables following  $x_1 \leftarrow \cosh(a_2 - a_0)^{-1}x_1$ ,  $x_2 \leftarrow \cosh(a_0 - a_1)^{-1}x_2$ , yields the announced formula.  $\blacksquare$

**Remark 3.1** It is important to mention that, in Theorem 3.1, the function spaces  $\mathcal{E}_\hbar$  are not invariant subspaces of  $C_\infty(R)$  under the action of  $R$ . In order to obtain invariant spaces, one can consider completions of the  $\mathcal{E}_\hbar$ 's with respect to suitable  $C^*$ -norms. This point has been treated in details in [4].

## 4 The case $n = 1$

In [4], one finds a strict quantization of the symplectic symmetric space  $M = (SO(1, 1) \times \mathbb{R}^2)/\mathbb{R}$ . Let us first briefly recall how this quantization is defined. It turns out that the above mentioned symplectic symmetric space is, as a symplectic manifold, globally symplectomorphic to the two-dimensional symplectic vector space  $(\mathbb{R}^2 = \{(a, l)\}, da \wedge dl)$ . With respect to the coordinate system  $(a, l)$ , the geodesic symmetries are

$$s_{(a, l)}(a', l') = (2a - a', 2 \cosh(a - a')l - l').$$

The curvature endomorphism of the underlying connection,  $\nabla$ , is given by

$$R(\partial_a, \partial_l) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Defining the following family of diffeomorphisms :

$$\phi_\hbar : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : \phi_\hbar(a, \alpha) = (a, \frac{2}{\hbar} \sinh(\frac{\hbar}{2}\alpha)),$$

a result analogous to Lemma 3.1 allows us to define a linear injection

$$\tau_\hbar : \mathcal{S}(\mathbb{R}^2) \rightarrow \mathcal{S}'(\mathbb{R}^2)$$

by

$$\tau_{\hbar} = F^{-1} \circ (\phi_{\hbar}^{-1})^{\star} \circ F,$$

where

$$Fu(a, \alpha) = \int_{\mathbb{R}} u(a, l) e^{-i\alpha l} dl.$$

Entirely similarly as in Section 3, the image space

$$\mathcal{E}_{\hbar} = \tau_{\hbar}(\mathcal{S}(\mathbb{R}^2)) \subset \mathcal{S}'(\mathbb{R}^2)$$

is shown to be endowed with an associative product defined by

$$a \star_{\hbar} b = \tau_{\hbar}(T_{\hbar} a \star_{\hbar}^W T_{\hbar} b),$$

where  $T_{\hbar} : \mathcal{S}(\mathbb{R}^2) \rightarrow \mathcal{S}(\mathbb{R}^2)$  is given by  $T_{\hbar} = F^{-1} \circ \phi_{\hbar}^{\star} \circ F$ , and where  $\star_{\hbar}^W$  denotes the Weyl product on  $\mathbb{R}^2$  (cf. Formula (9)). A computation similar (but simpler) to the one in the proof of Theorem 3.1 leads us to the following formula

$$u \star_{\hbar} v(x_0) = \frac{1}{\hbar^4} \int_{M \times M} e^{\frac{2i}{\hbar} S(x_0, x_1, x_2)} \cosh(a_1 - a_2) u(x_1) v(x_2) dx_1 dx_2 \quad (15)$$

for  $u, v \in \mathcal{S}(\mathbb{R}^2) \subset \mathcal{E}_{\hbar}$ , where  $x_i = (a_i, l_i) \in M = \mathbb{R}^2$ , where  $dx_i$  denotes the Liouville measure on  $M$  and where

$$S(x_0, x_1, x_2) = \oint_{0,1,2} \sinh(a_0 - a_1) l_2 \quad (x_i = (a_i, l_i) \in M = \mathbb{R}^2).$$

Besides associativity, the main property of the product  $\star_{\hbar}$  is its invariance under the transvection group  $G = SO(1, 1) \times \mathbb{R}^2$  of the symmetric space  $(M, \nabla)$ . In other words, both amplitude  $\cosh(a_1 - a_2)$  and phase  $S$  are invariant functions under the diagonal action of  $G$ .

Now, we observe that the transvection group  $SO(1, 1) \times \mathbb{R}^2$  actually contains a subgroup  $R$  isomorphic to the Iwasawa subgroup  $AN$  of  $SU(1, 1)$ . Indeed, the table of the Lie algebra  $\mathcal{G}$  of  $G$  is

$$\begin{aligned} [a, l] &= k \\ [k, a] &= -l \\ [k, l] &= 0. \end{aligned}$$

Therefore,  $\mathcal{R} = \text{span}\{a, k + l\}$  is a subalgebra isomorphic to  $\mathcal{A} \times \mathcal{N}$  in  $\text{su}(1, 1)$ . Observe that the analytic subgroup  $R$  of  $G$  with algebra  $\mathcal{R}$  acts simply transitively on  $M$ . Hence, the quantization  $\star_{\hbar}$  (cf. Formula (15)) defines a left-invariant strict quantization of the (symplectic) Lie group  $R$ . In particular, one can interpret Formula (15) in two ways. One way is to say that it is a degeneracy of Formula (14) for a one-dimensional nilpotent factor  $\mathcal{N}$ . This emphasizes more the group representation theoretical aspect of the construction. The other way relies on the fact that the phase  $S$  (as well as the amplitude) is determined uniquely in terms of the symmetric symplectic geometry of the symplectic symmetric space  $(M, \omega, \nabla)$  [16, 4].

We end this section by mentioning an equivalence between our product formula (15) in the degenerated case  $n = 1$  and Unterberger's formula for the composition of symbols in the one-dimensional Klein-Gordon Calculus (Formula (2.9) in [15], see also [14]). More precisely, let  $f_1$  and  $f_2 \in C_c^{\infty}(\mathbb{R}^2)$  be two compactly supported functions and let  $f_1 \sharp_{\hbar}^U f_2$  denote the symbol of the composition  $Op(f_1) \circ Op(f_2)$  in the one-dimensional Klein-Gordon Calculus (see [15] pp. 174). Define the diffeomorphism

$$\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : \varphi(a, l) = \left( \frac{1}{\cosh(a)} l, \sinh(a) \right).$$

Then, one has

$$\varphi^*(f_1 \sharp_{\hbar}^U f_2) = (\varphi^* f_1) \star_{\hbar} (\varphi^* f_2),$$

where  $\star_{\hbar}$  is the product defined in Formula (15).



## 5 Remark for further developments

Formulae (14) and (15) define left invariant associative multiplications on the spaces  $\mathcal{E}_\hbar$ 's. The latter spaces play an analogous role the Schwartz space  $\mathcal{S}(\mathcal{R})$  does in the case of Weyl's quantization. Each algebra  $(\mathcal{E}_\hbar, \star_\hbar)$  is isomorphic (via the "equivalence"  $T_\hbar$ ) to  $(\mathcal{S}(\mathcal{R}), \star_\hbar^W)$  (see Theorem 3.1). It would therefore not be surprising that the deformed products (14) and (15) extend to the space of smooth bounded functions, as in the case of Weyl's quantization [12]. This should provide actual universal deformations for any action of the group  $R$  on any  $C^*$ -algebra.

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